

# Mutation of torsion pairs in triangulated categories and its geometric realization

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## Abstract

By generalizing mutation of rigid subcategories, maximal rigid subcategories and cluster tilting subcategories, the notion of mutation of torsion pairs in triangulated categories is introduced. It is proved that the mutation of torsion pairs in triangulated categories are torsion pairs. It is also proved that there is no non-trivial mutation of t-structures, but shift. A geometric realization of mutation of torsion pairs in the cluster categories of type  $A_n$  or in the cluster categories of type  $A_\infty$  is given via the mutations (generalized flips) of Ptolemy diagrams of a regular  $(n + 3)$ -gon  $P_{n+3}$  or of a  $\infty$ -gon  $P_\infty$  respectively.

**Key words.** Torsion pairs, mutations, Ptolemy diagrams, triangulated categories.

**Mathematics Subject Classification.** 16G20, 16G70, 05A15, 13F60, 18E30.

## 1 Introduction

The notion of torsion pairs (or torsion theory) in abelian categories was introduced by Dickson [D]. It is important in algebra and geometry [BR] and plays an important role in representation theory of algebras, in particular in tilting theory [ASS]. The triangulated version of torsion pairs was introduced by Iyama and Yoshino [IY] in their study of mutation of cluster tilting subcategories in triangulated categories, see also [KR][BR]. Recently Ng gives a classification of torsion pairs in the cluster categories of  $A_\infty$  [Ng], Holm-Jørgensen-Rubery do the same for the cluster categories of  $A_n$  [HJR].

Cluster categories were introduced in [BMRRT], see also [CCS] for cluster categories of type  $A_n$ . They are the orbit categories  $D^b(\mathcal{H})/\tau^{-1}[1]$  of derived categories of hereditary categories  $\mathcal{H}$  arising from the action of subgroup  $\langle \tau^{-1}[1] \rangle$  of the automorphism group and are 2-Calabi-Yau triangulated categories [Ke1]. The aim for introducing these categories is to categorify cluster algebras. Cluster algebras were introduced by Fomin-Zelevinsky [FZ] in order to give an algebraic and combinatorial framework for the positivity and canonical basis of quantum groups. Cluster category and cluster tilting subcategories in cluster category, or more general in a 2-Calabi-Yau triangulated category have much nice properties. In particular, one can mutate cluster tilting objects, i.e. one can replace one indecomposable direct summand by a new indecomposable object got via a special triangle to obtain a new cluster tilting object. In cluster categories, the mutation of cluster tilting objects model the mutation of clusters of the corresponding cluster algebras. See the nice surveys [Ke2][Ke3][Re].

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Cluster tilting subcategories in triangulated categories are the torsion classes of certain torsion pairs. In general, a triangulated category (even a 2-Calabi-Yau triangulated category) may not admit any cluster tilting subcategory [KZ][BIRS]. In contrast, they always admit torsion pairs, for example, the trivial torsion pair: (the whole category, the zero category).

In this paper, we define and study the mutation of torsion pairs in triangulated categories. Under a reasonable condition on a subcategory  $\mathcal{D}$  in a fixed triangulated category  $\mathcal{C}$ , we show that the  $\mathcal{D}$ -mutation of a torsion pair  $(\mathcal{X}, \mathcal{Y})$  in  $\mathcal{C}$  is also a torsion pair  $(\mathcal{X}', \mathcal{Y}')$ , where the new torsion class  $\mathcal{X}'$  is the  $\mathcal{D}$ -mutation of  $\mathcal{X}$ , and the new torsion free class  $\mathcal{Y}'$  is the  $\mathcal{D}[1]$ -mutation of  $\mathcal{Y}$ . In the study of mutation of torsion pairs, the core of the torsion pair  $(\mathcal{X}, \mathcal{Y})$  which is defined as the subcategory  $\mathcal{X} \cap \mathcal{Y}[-1]$  plays an important role. Some properties of mutations of torsion pairs are given. Using the notion of Ptolemy diagrams defined in [HJR] and in [Ng], we give a geometric realization of mutation of torsion pairs in the cluster categories of type  $A_n$  and in the cluster categories of type  $A_\infty$ .

This paper is organized as follows: In section 2, some basic definitions and results are recalled. In any torsion pair  $(\mathcal{X}, \mathcal{Y})$ , we introduce the Ext-injective subcategories in  $\mathcal{X}$  (or Ext-projective subcategories in  $\mathcal{Y}$ ). They are called the core of the torsion pair, and are important in our study. In Section 3, fixed a subcategory  $\mathcal{D}$  which is functorially finite and rigid, and  $\tau[-1]\mathcal{D} = \mathcal{D}$ , the  $\mathcal{D}$ -mutation of torsion pairs is defined. It is proved that the  $\mathcal{D}$ -mutation of torsion pairs is also torsion pair, in which the cores of the corresponding torsion pairs form also a  $\mathcal{D}$ -mutation pair. A direct application to mutations of rigid subcategories, maximal rigid subcategories, and cluster tilting subcategories is given. It is also proved that the only possible mutation of t-structures is the shift of t-structures. We study the property of  $\mathcal{D}$ -mutation of torsion pairs. In section 4, we define the mutation of Ptolemy diagrams in a regular  $n+3$ -gon  $P_{n+3}$  and  $\infty$ -gon  $P_\infty$  respectively. It is proved that the mutation of Ptolemy diagrams in  $P_{n+3}$  or  $P_\infty$  coincides with the mutation of corresponding torsion pairs in the cluster category of type  $A_n$  or in the cluster category of type  $A_\infty$ , respectively.

## 2 Torsion pairs in triangulated categories

In this section we recall some basics on torsion pairs in a triangulated category. We first fix some notations. Let  $\mathcal{C}$  be an additive category. We write  $X \in \mathcal{C}$  to mean that  $X$  is an object of  $\mathcal{C}$ . For a subcategory  $\mathcal{X}$  of  $\mathcal{C}$ , we always assume that  $\mathcal{X}$  is a full subcategory and closed under taking isomorphisms, direct summands and finite direct sums. When  $\mathcal{X}$  is a subcategory, we use  $\mathcal{X}^\perp$  to denote the subcategory consisting of objects  $Y$  satisfying  $\text{Hom}_{\mathcal{C}}(X, Y) = 0$  for any  $X \in \mathcal{X}$ , and  ${}^\perp \mathcal{X}$  to denote the subcategory consisting of objects  $Y$  satisfying  $\text{Hom}_{\mathcal{C}}(Y, X) = 0$  for any  $X \in \mathcal{X}$ . For two subcategories  $\mathcal{X}, \mathcal{Y}$ , we write  $\text{Hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{Y}) = 0$  to mean that  $\text{Hom}_{\mathcal{C}}(X, Y) = 0$ , for any  $X \in \mathcal{X}$  and any  $Y \in \mathcal{Y}$ .

Now we assume that  $\mathcal{C}$  is a triangulated category with shift functor  $[1]$ . We denote by  $\text{Ext}^n(X, Y)$  the Hom space  $\text{Hom}_{\mathcal{C}}(X, Y[n])$ . We will use  $\text{Hom}(X, Y)$  to denote  $\text{Hom}_{\mathcal{C}}(X, Y)$  for simplicity if there is no confusion arisen. For two subcategories  $\mathcal{X}, \mathcal{Y}$ , we use  $\mathcal{X} * \mathcal{Y}$  to denote the subclass of  $\mathcal{C}$  consisting of objects  $Z$  such that there is a triangle  $X \rightarrow Z \rightarrow Y \rightarrow X[1]$  with  $X \in \mathcal{X}, Y \in \mathcal{Y}$ . It is easy to see that  $\mathcal{X} * \mathcal{Y}$  is closed under taking isomorphisms and finite direct sums. We call a subcategory  $\mathcal{X}$  is closed under extensions (or an extension-closed subcategory) if  $\mathcal{X} * \mathcal{X} \subset \mathcal{X}$ . If  $\text{Hom}(\mathcal{X}, \mathcal{Y}) = 0$ , then  $\mathcal{X} * \mathcal{Y}$  is closed under taking direct summands [IY]. In this case we understand  $\mathcal{X} * \mathcal{Y}$  as a subcategory of  $\mathcal{C}$ . We recall the definition of torsion pairs in a triangulated

category  $C$  from [IY].

**Definition 2.1.** Let  $C$  be a triangulated category. The pair  $(\mathcal{X}, \mathcal{Y})$  of subcategories of  $C$  is called a torsion pair if

$$\text{Hom}(\mathcal{X}, \mathcal{Y}) = 0 \text{ and } C = \mathcal{X} * \mathcal{Y}.$$

A pair  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair if and only if  $(\mathcal{X}, \mathcal{Y}[-1])$  is a cotorsion pair in the sense in [N]. A pair  $(\mathcal{M}, \mathcal{N})$  is called a cotorsion pair if  $\text{Ext}^1(\mathcal{M}, \mathcal{N}) = 0$  and  $C = \mathcal{M} * \mathcal{N}[1]$ . If  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair, then  $\mathcal{X} = {}^\perp \mathcal{Y}$ ,  $\mathcal{Y} = \mathcal{X}^\perp$ . It follows that  $\mathcal{X}$  (or  $\mathcal{Y}$ ) is a contravariantly (covariantly, respectively) finite and extension-closed subcategory of  $C$ . We call a subcategory  $\mathcal{X}$  is contravariantly finite in  $C$ , if any object  $M \in C$  admits a right  $\mathcal{X}$ -approximation  $f : X \rightarrow M$ , which means that any map from  $X' \in \mathcal{X}$  to  $M$  factors through  $f$ . The left  $\mathcal{X}$ -approximation of  $M$  and covariantly finiteness of  $\mathcal{X}$  can be defined dually.  $\mathcal{X}$  is called functorially finite in  $C$  if  $\mathcal{X}$  is both covariantly finite and contravariantly finite in  $C$ .

**Remark 2.2.** Let  $\mathcal{X}$  be a subcategory of  $C$  closed under taking extensions. Then  $(\mathcal{X}, \mathcal{X}^\perp)$  is a torsion pair if and only if  $\mathcal{X}$  is contravariantly finite in  $C$  [IY]. In particular, if  $\mathcal{X}$  is a contravariantly finite subcategory satisfying  $\text{Hom}(\mathcal{X}, \mathcal{X}[1]) = 0$ , then  $(\mathcal{X}, \mathcal{X}^\perp)$  is a torsion pair. This is because that for a subcategory  $\mathcal{X}$  with property  $\text{Hom}(\mathcal{X}, \mathcal{X}[1]) = 0$ ,  $\mathcal{X}$  is automatically closed under taking extensions. Dually, if  $\mathcal{X}$  is a covariantly finite subcategory satisfying  $\text{Hom}(\mathcal{X}, \mathcal{X}[1]) = 0$ , then  $({}^\perp \mathcal{X}, \mathcal{X})$  is a torsion pair.

**Definition 2.3.** Let  $\mathcal{X}$  be an extension-closed subcategory of  $C$ . The object  $T \in \mathcal{X}$  is called an Ext-injective object of  $\mathcal{X}$  provided  $\text{Ext}^1(\mathcal{X}, T) = 0$ . The Ext-projective object of  $\mathcal{X}$  is defined dually. The subcategory of  $\mathcal{X}$  consisting of Ext-injective (or Ext-projective) objects in  $\mathcal{X}$  is denoted by  $I(\mathcal{X})$  ( $P(\mathcal{X})$ , respectively).

**Proposition 2.4.** Let  $(\mathcal{X}, \mathcal{Y})$  be a torsion pair of  $C$ . Then  $I(\mathcal{X}) = \mathcal{X} \cap \mathcal{Y}[-1]$ ,  $P(\mathcal{Y}) = I(\mathcal{X})[1]$ . Moreover  $I(\mathcal{X})$  is covariantly finite in  $\mathcal{X}$ ,  $P(\mathcal{Y})$  is contravariantly finite in  $\mathcal{Y}$ .

*Proof.* We prove  $I(\mathcal{X}) = \mathcal{X} \cap \mathcal{Y}[-1]$ , the proof of  $P(\mathcal{Y}) = I(\mathcal{X})[1]$  is similar. Let  $T \in \mathcal{X}$ . Then  $\text{Ext}^1(\mathcal{X}, T) = \text{Hom}(\mathcal{X}, T[1]) = 0$  if and only if  $T[1] \in \mathcal{Y}$ . Then  $T \in I(\mathcal{X})$  if and only if  $T \in \mathcal{X} \cap \mathcal{Y}[-1]$ .

Now we prove that  $I(\mathcal{X})$  is covariantly finite in  $\mathcal{X}$ , the proof for the contravariantly finiteness of  $P(\mathcal{Y})$  in  $\mathcal{Y}$  is similar. Since  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair, we have that  $(\mathcal{X}[-1], \mathcal{Y}[-1])$  is a torsion pair. Then for any  $X \in \mathcal{X}$ , there exists a triangle  $X_1[-1] \rightarrow X \xrightarrow{g} Y[-1] \rightarrow X_1$  with  $X_1 \in \mathcal{X}$  and  $Y \in \mathcal{Y}$ . It follows  $Y[-1] \in \mathcal{X}$ , and hence  $Y[-1] \in I(\mathcal{X})$ . It is easy to see that  $g$  is a left  $I(\mathcal{X})$ -approximation of  $X$ . Then  $I(\mathcal{X})$  is a covariantly finite in  $\mathcal{X}$ .  $\square$

**Definition 2.5.** Let  $(\mathcal{X}, \mathcal{Y})$  be a torsion pair in  $C$ . We call  $I(\mathcal{X})$  the core of the torsion pair  $(\mathcal{X}, \mathcal{Y})$ , which is usually denoted by  $I$ . In this case we also call  $I$  the core of the corresponding cotorsion pair  $(\mathcal{X}, \mathcal{Y}[1])$ .

There are several special cases of torsion pairs, which are received much attention recently [BIRS, BMV, BR, IY, KR, KZ, MP, N]:

**Definition 2.6.** Let  $\mathcal{X}, \mathcal{Y}$  be subcategories of a triangulated category  $C$ .

1. The pair  $(\mathcal{X}, \mathcal{Y})$  is called a  $t$ -structure in  $\mathcal{C}$  if  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair and  $\mathcal{X}$  is closed under  $[1]$  (equivalently  $\mathcal{Y}$  is closed under  $[-1]$ ). In this case  $\mathcal{X} \cap \mathcal{Y}[1]$  is an abelian category, which is called the heart of  $(\mathcal{X}, \mathcal{Y})$  [BBD][BR].
2. The pair  $(\mathcal{X}, \mathcal{Y})$  is called a rigid torsion pair if  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair and  $\text{Ext}^1(\mathcal{X}, \mathcal{X}) = 0$ . In this case, we call the subcategory  $\mathcal{X}$  a rigid subcategory [BIRS].
3. The pair  $(\mathcal{X}, \mathcal{Y})$  is called a cluster tilting torsion pair if  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair and satisfies the property:  $X \in \mathcal{X}$  if and only if  $\text{Ext}^1(\mathcal{X}, X) = 0$ . In this case, we call the subcategory  $\mathcal{X}$  a cluster tilting subcategory [BMRRT].
4. The pair  $(\mathcal{X}, \mathcal{Y})$  is called a maximal rigid torsion pair provided  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair,  $\mathcal{X}$  is rigid and satisfies the property: "if  $\text{Ext}^1(M \oplus X, M \oplus X') = 0$  for any  $X, X' \in \mathcal{X}$ , then  $M \in \mathcal{X}$ ". In this case, we call the subcategory  $\mathcal{X}$  a maximal rigid subcategory [BIRS][BMV].

**Remark 2.7.** The origin definition of cluster tilting subcategories [KR] requires more conditions than that in 3. in Definition 2.6. It turns out by Lemma 3.2 in [KZ] that the present one is equivalent to the origin one.

**Remark 2.8.** The cotorsion pair  $(\mathcal{X}, \mathcal{W})$  is called a rigid cotorsion pair (a cluster tilting cotorsion pair; or a maximal rigid cotorsion pair respectively) if the torsion pair  $(\mathcal{X}, \mathcal{W}[1])$  is a rigid torsion pair (cluster tilting torsion pair or a maximal rigid torsion pair respectively).

It is more common to define a  $t$ -structure to be a pair  $(\mathcal{T}^{\leq 0}, \mathcal{T}^{\geq 0})$  of subcategories satisfying a short list of conditions; see for example [BBD][BR]. This definition is equivalent to the one above if we put  $\mathcal{X} = \mathcal{T}^{\leq 0}$  and  $\mathcal{Y} = \mathcal{T}^{\geq 0}[-1]$ . The subcategory  $\mathcal{X}$  is sometimes referred to as an aisle [KeV]. From definition, A torsion pair can be regarded as a  $t$ -structure without the shift-closedness.

**Proposition 2.9.** Let  $(\mathcal{X}, \mathcal{Y})$  be a torsion pair in  $\mathcal{C}$  and  $I$  its core. Then

1.  $(\mathcal{X}, \mathcal{Y})$  is a  $t$ -structure in  $\mathcal{C}$  if and only if  $I = 0$ .
2.  $(\mathcal{X}, \mathcal{Y})$  is a rigid torsion pair if and only if  $I = \mathcal{X}$ .
3.  $(\mathcal{X}, \mathcal{Y})$  is a cluster tilting torsion pair if and only if  $\mathcal{Y} = \mathcal{X}[1]$ .
4.  $(\mathcal{X}, \mathcal{Y})$  is a maximal rigid torsion pair if and only if  $\mathcal{X}[1] \subseteq \mathcal{Y}$ , and any rigid object belongs to  $\mathcal{X} * \mathcal{X}[1]$ .

*Proof.* 1. If  $(\mathcal{X}, \mathcal{Y})$  is a  $t$ -structure in  $\mathcal{C}$  then  $\mathcal{Y}[-1] \subseteq \mathcal{Y}$ . Hence  $I = \mathcal{X} \cap \mathcal{Y}[-1] \subseteq \mathcal{X} \cap \mathcal{Y} = 0$ .

Now suppose that  $I = 0$ . For any  $Y[-1] \in \mathcal{Y}[-1]$ , we have a triangle  $X \rightarrow Y[-1] \rightarrow Y_1 \rightarrow X[1]$  with  $X \in \mathcal{X}$ , and  $Y_1 \in \mathcal{Y}$ . It follows that  $X[1] \in \mathcal{Y}$ , and subsequently  $X \in \mathcal{Y}[-1]$ . Then  $X = 0$  due to that  $X \in I = 0$ , and hence we have that  $Y[-1] \cong Y_1 \in \mathcal{Y}$ . This proves that  $\mathcal{Y}[-1] \subseteq \mathcal{Y}$ . Therefore  $(\mathcal{X}, \mathcal{Y})$  is a  $t$ -structure.

2. It follows directly from the definition.

3. It follows directly from the definition.
4. The one direction was proved in Corollary 2.5 in [ZZ], we prove the other direction. Let  $(\mathcal{X}, \mathcal{Y})$  be a torsion pair with  $\mathcal{X}[1] \subseteq \mathcal{Y}$ , and any rigid object belongs to  $\mathcal{X} * \mathcal{X}[1]$ . Then  $\mathcal{X}$  is rigid. If we have an object  $M$  satisfies  $\text{Ext}^1(M \oplus X_1, M \oplus X_2) = 0$  for any  $X_1, X_2 \in \mathcal{X}$ , then  $M$  is rigid. It follows that there is a triangle  $X \rightarrow M \rightarrow X' \rightarrow X[1]$ , where  $X, X' \in \mathcal{X}$ . Then the triangle splits, and  $M \simeq X \oplus X' \in \mathcal{X}$ .  $\mathcal{X}$  is maximal rigid.  $\square$

### 3 Mutation of torsion pairs

Let  $\mathcal{C}$  be a triangulated category and  $\mathcal{D}$  a rigid subcategory of  $\mathcal{C}$ , i.e.  $\text{Ext}^1(\mathcal{D}, \mathcal{D}) = 0$ . For any subcategory  $\mathcal{M} \supset \mathcal{D}$ , put:

$$\mu^{-1}(\mathcal{M}; \mathcal{D}) := (\mathcal{D} * \mathcal{M}[1]) \cap {}^\perp(\mathcal{D}[1]).$$

Dually, for a subcategory  $\mathcal{N} \supset \mathcal{D}$ , put:

$$\mu(\mathcal{N}; \mathcal{D}) := (\mathcal{N}[-1] * \mathcal{D}) \cap (\mathcal{D}[-1])^\perp.$$

The notion of  $\mathcal{D}$ -mutation is defined in [IY] as a generalization of mutation of cluster tilting objects in cluster categories.

**Definition 3.1.** *The pair  $(\mathcal{M}, \mathcal{N})$  of subcategories of  $\mathcal{C}$  is called a  $\mathcal{D}$ -mutation pair if  $\mathcal{M} = \mu(\mathcal{N}; \mathcal{D})$  and  $\mathcal{N} = \mu^{-1}(\mathcal{M}; \mathcal{D})$ .*

It is not difficult to see that: for subcategories  $\mathcal{M}, \mathcal{N}$ , both containing  $\mathcal{D}$ ,  $(\mathcal{M}, \mathcal{N})$  forms a  $\mathcal{D}$ -mutation if and only if for any  $X \in \mathcal{M}$ , there is a triangle  $X \xrightarrow{f} \mathcal{D} \xrightarrow{g} Y \rightarrow X[1]$  where  $\mathcal{D} \in \mathcal{D}$ ,  $Y \in \mathcal{N}$  and  $f$  (or  $g$ ) is a left (right, respectively)  $\mathcal{D}$ -approximation; and for any  $Y \in \mathcal{N}$ , there is a triangle  $X \xrightarrow{f'} \mathcal{D} \xrightarrow{g'} Y \rightarrow X[1]$  where  $\mathcal{D} \in \mathcal{D}$ ,  $X \in \mathcal{M}$  and  $g'$  (or  $f'$ ) is a right (left, respectively)  $\mathcal{D}$ -approximation. In this case, we write  $Y = \mu_{\mathcal{D}}^{-1}(X)$  and  $X = \mu_{\mathcal{D}}(Y)$ . We have that  $\mu^{-1}(\mathcal{M}; \mathcal{D})$  is the subcategory of  $\mathcal{C}$  generated (additively) by  $\mu_{\mathcal{D}}^{-1}(X)$ ,  $X \in \mathcal{M}$  and  $\mathcal{D}$ ,  $\mu(\mathcal{N}; \mathcal{D})$  is the subcategory of  $\mathcal{C}$  generated (additively) by  $\mu_{\mathcal{D}}(Y)$ ,  $Y \in \mathcal{N}$  and  $\mathcal{D}$ .

We note that if  $\mathcal{D} = 0$ , then  $\mu^{-1}(\mathcal{M}; \mathcal{D}) = \mathcal{M}[1]$  and  $\mu(\mathcal{N}; \mathcal{D}) = \mathcal{N}[-1]$ .

Set  $\check{\mathcal{D}} := \{X \in \mathcal{C} \mid \exists \text{ a left } \mathcal{D} \text{ - approximation } f : X \rightarrow \mathcal{D}\}$

$\check{\mathcal{D}}$  is a subcategory of  $\mathcal{C}$  closed under direct summands and finite direct sums,  $\check{\mathcal{D}} \supseteq \mathcal{D}$ .

Dually, we set  $\hat{\mathcal{D}} := \{Y \in \mathcal{C} \mid \exists \text{ a right } \mathcal{D} \text{ - approximation } f : \mathcal{D} \rightarrow Y\}$  which is also a subcategory of  $\mathcal{C}$  closed under direct summands and finite direct sums,  $\hat{\mathcal{D}} \supseteq \mathcal{D}$ .

Note that  $\mathcal{D}$  is contravariantly finite if and only if  $\hat{\mathcal{D}} = \mathcal{C}$ , and dually  $\mathcal{D}$  is covariantly finite if and only if  $\check{\mathcal{D}} = \mathcal{C}$ . Thus  $\mathcal{D}$  is functorially finite in  $\mathcal{C}$  if and only if  $\hat{\mathcal{D}} = \mathcal{C}$  and  $\check{\mathcal{D}} = \mathcal{C}$ .

The following result was proved in [IY] for the case that  $\mathcal{D}$  is functorially finite. But their proof can be applied for the general case without any change.

**Proposition 3.2.** *Let  $\mathcal{D}$  be a subcategory of  $\mathcal{C}$  satisfying  $\text{Hom}(\mathcal{D}, \mathcal{D}[1]) = 0$ . Then  $\mathcal{M} \mapsto \mu^{-1}(\mathcal{M}; \mathcal{D})$  gives a one-to-one correspondence between the set of subcategories  $\mathcal{M}$  of  $\mathcal{C}$  satisfying  $\mathcal{D} \subset \mathcal{M} \subset \check{\mathcal{D}} \cap (\mathcal{D}[-1])^\perp$  and the set of subcategories  $\mathcal{N}$  of  $\mathcal{C}$  satisfying  $\mathcal{D} \subset \mathcal{N} \subset \hat{\mathcal{D}} \cap {}^\perp(\mathcal{D}[1])$ . The inverse is given by  $\mathcal{N} \mapsto \mu(\mathcal{N}; \mathcal{D})$ .*

*Proof.* See the proof of Proposition 2.7 in [IY].  $\square$

From now to the end of this section in this paper, we assume that  $C$  is a triangulated category with Serre duality which we recall now. Fix an algebraically closed field  $k$ . A triangulated category  $C$  is called  $k$ -linear provided all Hom-spaces in  $C$  are  $k$ -spaces and the compositions of maps are  $k$ -linear. The  $k$ -linear triangulated categories in this paper will be assumed Hom-finite and Krull-Remak-Schmidt, i.e.  $\dim_k \text{Hom}(X, Y) < \infty$  for any two objects  $X$  and  $Y$  in  $C$ , and every object decomposes into a finite direct sum of objects having local endomorphism rings [H].

A functor  $\Sigma : C \rightarrow C$  is called a Serre functor if there is a bifunctorial isomorphism  $\text{Hom}(X, Y) \cong D\text{Hom}(Y, \Sigma X)$  for objects  $X, Y \in C$ , where  $D = \text{Hom}_k(-, k)$  [Ke1]. In this case,  $C$  is called a triangulated category with Serre duality. A  $k$ -linear triangulated category  $C$  is called 2-Calabi-Yau (2-CY for short) if the functor  $[2]$  is the Serre functor [Ke1][Ke2].

If  $C$  has Serre functor  $\Sigma$ , then it has Auslander-Reiten triangles and the Auslander-Reiten translate  $\tau$ . We have that  $\Sigma = \tau[1]$  [RV].

In what follows, we always assume that the subcategory  $\mathcal{D}$  of  $C$  satisfies the following condition **RF** :  $\mathcal{D}$  is a functorially finite rigid subcategory, and  $\mathcal{D}$  is  $F_2$ -subcategory, i.e.  $F_2\mathcal{D} = \mathcal{D}$  (equivalently  $\tau\mathcal{D} = \mathcal{D}[1]$ ), where  $F_2 = \Sigma^{-1}[2]$ .

We note that if  $C$  is a 2-CY triangulated category, then  $F_2 = id$ , and subsequently any functorially finite rigid subcategory  $\mathcal{D}$  satisfies the condition **RF**.

Under this condition,  ${}^\perp\mathcal{D}[1] = \mathcal{D}[-1]^\perp$ , which is denoted by  $\mathcal{Z}$ . It is easy to see that  $(\mathcal{Z}, \mathcal{D})$  forms a  $\mathcal{D}$ -mutation pair [IY]. The quotient category  $\mathcal{U} := \mathcal{Z}/\mathcal{D}$  is defined as follows: the objects are the same as  $\mathcal{Z}$ , the Hom space from  $X$  to  $Y$  is defined as the quotient group of  $\text{Hom}_C(X, Y)$  by the subgroup consisting of morphisms factoring through an object in  $\mathcal{D}$ . It was proved by Iyama-Yoshino that the quotient category  $\mathcal{U}$  carries a natural triangulated structure inherited from the triangulated structure of  $C$  (see Proposition 4.6 and Theorem 4.2 in [IY]). For the convenience of the reader, We recall briefly the triangulated structure of  $\mathcal{Z}/\mathcal{D}$  below:

The shift in  $\mathcal{U}$  is defined as follows: for any object  $X$ , consider the left  $\mathcal{D}$ -approximation  $f : X \rightarrow D$ , and extend it to a triangle  $X \xrightarrow{f} D \rightarrow Z \rightarrow X[1]$ , then  $Z$  is defined as the shift of  $X$  in  $\mathcal{U}$ , denoted as:  $X < 1 >$ .

For any triangle  $X \xrightarrow{a} Y \xrightarrow{b} Z \xrightarrow{c} X[1]$  in  $C$ , take a triangle  $X \xrightarrow{f} D \xrightarrow{g} X < 1 > \xrightarrow{h} X[1]$  with  $f$  being the left  $\mathcal{D}$ -approximation of  $X$ . Then we have the commutative diagram:

$$\begin{array}{ccccccc} X & \xrightarrow{a} & Y & \xrightarrow{b} & Z & \xrightarrow{c} & X[1] \\ \parallel & & \downarrow & & \downarrow d & & \parallel \\ X & \xrightarrow{f} & D & \xrightarrow{g} & X < 1 > & \xrightarrow{h} & Y[1]. \end{array}$$

Then triangles in  $\mathcal{U}$  is defined as the complexes in  $\mathcal{U}$  which are isomorphic to a complex

$$X \xrightarrow{\bar{a}} Y \xrightarrow{\bar{b}} Z \xrightarrow{\bar{c}} X\langle 1 \rangle$$

where  $X, Y, Z \in \mathcal{Z}$ , and  $\bar{a}, \bar{b}, \bar{c}$  are the images of maps  $a, b, c$  under the quotient functor  $\mathcal{Z} \rightarrow \mathcal{U}$  respectively.

Under the condition **RF** on  $\mathcal{D}$ , when  $\mathcal{X}$  is a subcategory of  $C$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$ , we have the following result.

**Lemma 3.3.** *If  $(\mathcal{X}, \mathcal{W})$  is a cotorsion pair and  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$ , then  $\mathcal{D} \subset \mathcal{W} \subset \mathcal{D}[-1]^\perp$ .*

*Proof.* Note that  $\mathcal{W} = \mathcal{X}[-1]^\perp$ . Since  $\mathcal{X} \subset \mathcal{D}[-1]^\perp = {}^\perp\mathcal{D}[1]$ , we have  $\text{Hom}(\mathcal{X}, \mathcal{D}[1]) = 0$ , and then  $\text{Hom}(\mathcal{X}[-1], \mathcal{D}) = 0$ . Therefore,  $\mathcal{D} \subset \mathcal{X}[-1]^\perp = \mathcal{W}$ . Since  $\mathcal{D} \subset \mathcal{X}$ ,  $\mathcal{D}[-1] \subset \mathcal{X}[-1]$ . Hence  $\mathcal{X}[-1]^\perp \subset \mathcal{D}[-1]^\perp$ .  $\square$

Thus, one can consider the relations between cotorsion pairs in  $\mathcal{C}$  and cotorsion pairs in the quotient category  $\mathcal{U}$ . We shall need the following lemma.

**Lemma 3.4.** *Let  $X$  and  $W$  be two objects in  $\mathcal{Z}$ . Then  $\text{Hom}_{\mathcal{C}}(X, W[1]) = 0$  if and only if  $\text{Hom}_{\mathcal{U}}(X, W < 1 >) = 0$ .*

*Proof.* Since  $W \in \mathcal{Z}$ , there is a triangle in  $\mathcal{C}$

$$W \rightarrow D_W \xrightarrow{g} W < 1 > \rightarrow W[1]$$

where  $D_W \in \mathcal{D}$  and  $g$  is a right  $\mathcal{D}$ -approximation of  $W < 1 >$ . Applying the functor  $\text{Hom}_{\mathcal{C}}(X, -)$  to this triangle, we have the following exact sequence

$$\text{Hom}_{\mathcal{C}}(X, D_W) \rightarrow \text{Hom}_{\mathcal{C}}(X, W < 1 >) \rightarrow \text{Hom}_{\mathcal{C}}(X, W[1]) \rightarrow 0$$

as  $\text{Hom}_{\mathcal{C}}(X, \mathcal{D}[1]) = 0$ . It follows that the lemma holds.  $\square$

The following theorem gives a one-to-one correspondence between cotorsion pairs whose core containing  $\mathcal{D}$  in  $\mathcal{C}$  and cotorsion pairs in  $\mathcal{U}$ . In the following,  $\overline{\mathcal{X}}$  denotes the subcategory of  $\mathcal{U}$  consisting of objects  $X \in \mathcal{X}$ , for the subcategory  $\mathcal{X}$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$ .

**Theorem 3.5.** *Assume that  $\mathcal{X}$  is a subcategory of  $\mathcal{C}$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$ . Then  $(\mathcal{X}, \mathcal{W})$  is a cotorsion pair with the core  $I$  in  $\mathcal{C}$  if and only if  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a cotorsion pair with the core  $I(\mathcal{X})$  in  $\mathcal{U}$ .*

*Proof.* Firstly we note that  $\mathcal{U}$  is a triangulated category with shift functor  $< 1 >$ .

Assume that  $(\mathcal{X}, \mathcal{W})$  is a cotorsion pair. Then  $\text{Hom}_{\mathcal{U}}(X, W < 1 >) = 0$  by Lemma 3.4. For any  $Z \in \mathcal{Z}$ , there is a triangle

$$W \rightarrow X \rightarrow Z \rightarrow W[1]$$

where  $X \in \mathcal{X}$  and  $W \in \mathcal{W}$  as  $(\mathcal{X}, \mathcal{W})$  is a cotorsion pair in  $\mathcal{C}$ . Since all of  $W, X, Z$  are in  $\mathcal{Z}$ , there is a triangle  $W \rightarrow X \rightarrow Z \rightarrow W < 1 >$  in  $\mathcal{U}$ . Therefore,  $\mathcal{U} \subset \overline{\mathcal{X}} * \overline{\mathcal{W}} < 1 >$ , and then  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a cotorsion pair in  $\mathcal{U}$ .

Conversely, assume that  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a cotorsion pair. Then  $\text{Hom}_{\mathcal{C}}(\mathcal{X}, \mathcal{W}) = 0$  by Lemma 3.4. For any  $Z \in \mathcal{Z}$ , there is a triangle in  $\mathcal{U}$ :

$$W \rightarrow X \rightarrow Z \rightarrow W < 1 >$$

where  $X \in \mathcal{X}$  and  $W \in \mathcal{X}[-1]^\perp$ . Then there is a triangle

$$W \rightarrow X \rightarrow Z' \rightarrow W[1]$$

in  $\mathcal{C}$  such that  $Z \cong Z'$  in  $\mathcal{U}$ . Then  $Z \cong Z'$  in  $\mathcal{C}$  up to direct summands in  $\mathcal{D}$ . Thus  $\mathcal{Z}$  is a subcategory of  $\mathcal{X} * \mathcal{W}[1]$ . Then  $\mathcal{C} = \mathcal{Z} * \mathcal{D}[1] \subset \mathcal{X} * \mathcal{W}[1] * \mathcal{D}[1] = \mathcal{X} * \mathcal{W}[1]$ . Therefore  $(\mathcal{X}, \mathcal{W})$  is a cotorsion pair in  $\mathcal{C}$ .

Finally, we have that  $I(\mathcal{X}) = \overline{\mathcal{X}} \cap \overline{\mathcal{W}} = \overline{\mathcal{X} \cap \mathcal{W}} = \overline{I(\mathcal{X})}$ .  $\square$

Restricting ourselves to the special cases of cluster tilting torsion pairs, rigid torsion pairs, maximal rigid torsion pairs and t-structures, we have the following results.

**Corollary 3.6.** *Assume that  $\mathcal{X}$  is a subcategory of  $\mathcal{C}$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$ .*

1.  *$(\mathcal{X}, \mathcal{W})$  is a rigid cotorsion pair in  $\mathcal{C}$  if and only if  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a rigid cotorsion pair in  $\mathcal{U}$ .*
2.  *$(\mathcal{X}, \mathcal{W})$  is a cluster tilting cotorsion pair in  $\mathcal{C}$  if and only if  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a cluster tilting cotorsion pair in  $\mathcal{U}$ .*
3.  *$(\mathcal{X}, \mathcal{W})$  is a maximal rigid cotorsion pair in  $\mathcal{C}$  if and only if  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a maximal rigid cotorsion pair in  $\mathcal{U}$ .*
4.  *$(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a t-structure in  $\mathcal{U}$  if and only if  $\mathcal{D} = I(\mathcal{X})$ .*

*Proof.* 1. By Proposition 2.9, we only need to prove that  $I(\mathcal{X}) = \mathcal{X}$  if and only if  $I(\overline{\mathcal{X}}) = \overline{\mathcal{X}}$ . But by Theorem 3.5,  $I(\overline{\mathcal{X}}) = \overline{I(\mathcal{X})}$ . Thus the assertion holds.

2. It follows from Theorem 5.1 in [IY].

3. Obviously,  $\mathcal{X} \subset \mathcal{W}$  if and only if  $\overline{\mathcal{X}} \subset \overline{\mathcal{W}}$ . Suppose that any rigid object in  $\mathcal{C}$  belongs to  $\mathcal{X} * \mathcal{X}[1]$ . For any rigid object  $Y$  in  $\mathcal{U}$ , we know that  $Y$  is rigid in  $\mathcal{C}$  by the first part of this proposition. Then there is a triangle in  $\mathcal{C}$ :

$$X' \rightarrow X \rightarrow Y \rightarrow X'[1]$$

with  $X, X' \in \mathcal{X}$ . Then there is a triangle in  $\mathcal{U}$ :

$$X' \rightarrow X \rightarrow Y \rightarrow X' < 1 > .$$

Therefore,  $Y \in \overline{\mathcal{X}} * \overline{\mathcal{X}} < 1 >$ . It follows from Proposition 2.9 that  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a maximal rigid cotorsion pair in  $\mathcal{U}$ .

Conversely, suppose that any rigid object in  $\mathcal{U}$  belongs to  $\overline{\mathcal{X}} * \overline{\mathcal{X}} < 1 >$ . For any rigid object  $Y$  in  $\mathcal{C}$ , there is a triangle in  $\mathcal{C}$ :

$$Z \rightarrow Y \rightarrow D[1] \rightarrow Z[1]$$

with  $D \in \mathcal{D}$  and  $Z \in \mathcal{Z}$ . It is easily checked that  $Z$  is a rigid object in  $\mathcal{C}$ . Then it is rigid in  $\mathcal{U}$  by the first part of this proposition, and then there is a triangle in  $\mathcal{U}$ :

$$X' \rightarrow X \rightarrow Z \rightarrow X' < 1 > .$$

with  $X, X' \in \mathcal{X}$ . Then there is a triangle in  $\mathcal{C}$

$$X' \rightarrow X \rightarrow Z' \rightarrow X'[1].$$

such that  $Z'$  is isomorphic to  $Z$  up to direct summands in  $\mathcal{D}$ . Therefore,  $Y \in \overline{\mathcal{X}} * \overline{\mathcal{X}} < 1 >$ . Since  $\text{Hom}(\mathcal{X}, \mathcal{X}[1]) = 0$ , then  $\mathcal{X} * \mathcal{X}[1]$  is closed under extensions. Therefore  $Z \in \mathcal{X} * \mathcal{X}[1]$  and then  $Y \in \mathcal{X} * \mathcal{X}[1] * \mathcal{D}[1] = \mathcal{X} * \mathcal{X}[1]$ . By Proposition 2.9, we have that  $(\mathcal{X}, \mathcal{W})$  is a cotorsion pair in  $\mathcal{C}$ .



4. It follows from Proposition 2.9 and Theorem 3.5. □

The follow theorem is a generalization of well-known results on mutations of cluster tilting objects [BMRRT][IY], maximal rigid objects[BIRS][ZZ] to the setting of torsion pairs.

**Theorem 3.7.** *Assume that  $(\mathcal{X}, \mathcal{W})$  is a cotorsion pair with the core  $I$  in  $\mathcal{C}$ ,  $\mathcal{D}$  is a subcategory of  $I$  satisfying the condition **RF**. Then  $(\mu^{-1}(\mathcal{X}; \mathcal{D}), \mu^{-1}(\mathcal{W}; \mathcal{D}))$  is also a cotorsion pair in  $\mathcal{C}$ . Moreover  $\mu^{-1}(I; \mathcal{D}) = I(\mu^{-1}(\mathcal{X}; \mathcal{D}))$ , and  $(I, I(\mu^{-1}(\mathcal{X}; \mathcal{D})))$  forms a  $\mathcal{D}$ -mutation.*

*Proof.* We denote  $\mu^{-1}(\mathcal{X}; \mathcal{D}), \mu^{-1}(\mathcal{W}; \mathcal{D}), \mu^{-1}(I; \mathcal{D})$  by  $\mathcal{X}', \mathcal{W}', I'$  respectively.

By Theorem 3.5, we have that  $(\overline{\mathcal{X}}, \overline{\mathcal{W}})$  is a cotorsion pair in  $\mathcal{U}$ . Then its 0-mutation  $(\overline{\mathcal{X}} < 1 >, \overline{\mathcal{W}} < 1 >)$  is also cotorsion pair in  $\mathcal{U}$ . By Proposition 4.4 in [IY],  $\overline{\mathcal{X}} < 1 > = \overline{\mathcal{X}'}$  and  $\overline{\mathcal{W}} < 1 > = \overline{\mathcal{W}'}$ . Then by Theorem 3.5 again, we have that  $(\mathcal{X}', \mathcal{W}')$  is a cotorsion pair in  $\mathcal{C}$ .

For the last statement, we have that  $I(\overline{\mathcal{X}}) = \overline{I(\mathcal{X})}$  and  $I(\overline{\mathcal{X}'}) = \overline{I(\mathcal{X}'})$  by Theorem 3.5. It is easily checked that  $I(\overline{\mathcal{X}}) < 1 > = I(\overline{\mathcal{X}} < 1 >) = I(\overline{\mathcal{X}'}) = \overline{I(\mathcal{X}')}.$

$$\overline{I'} = \overline{I(\mathcal{X}')} < 1 > = I(\overline{\mathcal{X}}) < 1 > = I(\overline{\mathcal{X}} < 1 >) = I(\overline{\mathcal{X}'}) = \overline{I(\mathcal{X}')}.$$

Therefore,  $I' = I(\mathcal{X}')$  and  $(I, I(\mathcal{X}'))$  forms a  $\mathcal{D}$ -mutation. □

Note that  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair if and only if  $(\mathcal{X}, \mathcal{Y}[-1])$  is a cotorsion pair, and  $\mu^{-1}(\mathcal{Y}; \mathcal{D}[1])[-1] = \mu^{-1}(\mathcal{Y}[-1]; \mathcal{D})$ . Then we have the torsion pair version of the above theorem.

**Theorem 3.8.** *(The torsion pair version) Assume that  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair with the core  $I$  in  $\mathcal{C}$ ,  $\mathcal{D}$  is a subcategory of  $I$  satisfying the condition **RF**. Then  $(\mu^{-1}(\mathcal{X}; \mathcal{D}), \mu^{-1}(\mathcal{Y}; \mathcal{D}[1]))$  is also a torsion pair in  $\mathcal{C}$ . Moreover  $\mu^{-1}(I; \mathcal{D}) = I(\mu^{-1}(\mathcal{X}; \mathcal{D}))$ , and  $(I, I(\mu^{-1}(\mathcal{X}; \mathcal{D})))$  forms a  $\mathcal{D}$ -mutation.*

Dually, one can prove that  $(\mu(\mathcal{X}; \mathcal{D}), \mu(\mathcal{Y}; \mathcal{D}[1]))$  is a torsion pair if  $(\mathcal{X}, \mathcal{Y})$  is a torsion pair. It follows from Theorem 3.8 and its dual, Propositions 3.4, 3.7 that if  $(\mathcal{X}, \mathcal{Y}), (\mathcal{X}', \mathcal{Y}')$  are two torsion pairs in  $\mathcal{C}$ , then  $(\mathcal{X}', \mathcal{Y}') = (\mu^{-1}(\mathcal{X}; \mathcal{D}), \mu^{-1}(\mathcal{Y}; \mathcal{D}[1]))$  if and only if  $(\mathcal{X}, \mathcal{Y}) = (\mu(\mathcal{X}'; \mathcal{D}), \mu(\mathcal{Y}'; \mathcal{D}))$ .

**Definition 3.9.** *Let  $\mathcal{D}$  be a functorially finite rigid  $F_2$ -subcategory of  $\mathcal{C}$  and  $(\mathcal{X}, \mathcal{Y}), (\mathcal{X}', \mathcal{Y}')$  two torsion pairs in  $\mathcal{C}$  satisfying  $\mathcal{D} \subset \mathcal{X}, \mathcal{X}' \subset \mathcal{D}[-1]^\perp$ . We call the torsion pairs  $(\mu^{-1}(\mathcal{X}; \mathcal{D}), \mu^{-1}(\mathcal{Y}; \mathcal{D}[1])), (\mu(\mathcal{X}'; \mathcal{D}), \mu(\mathcal{Y}'; \mathcal{D}[1]))$  are the  $\mathcal{D}$ -mutations of torsion pairs  $(\mathcal{X}, \mathcal{Y}), (\mathcal{X}', \mathcal{Y}')$  respectively. We call the equivalence  $\approx_{\mathcal{D}}$  on the class of torsion pairs in  $\mathcal{C}$  generated by the  $\mathcal{D}$ -mutations of torsion pairs the  $\mathcal{D}$ -mutation equivalence. The equivalence class of a torsion pair  $(\mathcal{X}, \mathcal{Y})$  is denoted by  $(\mathcal{X}, \mathcal{Y})_{\mathcal{D}}$ .*

From the definition above, we have that two torsion pairs  $(\mathcal{X}, \mathcal{Y}), (\mathcal{X}', \mathcal{Y}')$  form a  $\mathcal{D}$ -mutation if and only if  $\mathcal{X}' = \mu^{-1}(\mathcal{X}; \mathcal{D})$  and  $\mathcal{Y}' = \mu^{-1}(\mathcal{Y}; \mathcal{D}[1])$  if and only if  $\mathcal{X} = \mu(\mathcal{X}'; \mathcal{D})$  and  $\mathcal{Y} = \mu(\mathcal{Y}'; \mathcal{D}[1])$ .

Now we apply the mutation of torsion pairs to the special cases of  $t$ -structures, cluster tilting torsion pairs, rigid torsion pairs, and maximal rigid torsion pairs. The mutations of these special cases appeared recently in the study of cluster tilting theory and cluster algebras, see the surveys [Ke3] [Re] and the references there.

**Corollary 3.10.** *Let  $\mathcal{D}$  be a functorially finite rigid  $F_2$ -subcategory of  $\mathcal{C}$ ,  $(\mathcal{X}, \mathcal{Y})$  a torsion pair in  $\mathcal{C}$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$  and  $(\mathcal{X}, \mathcal{Y})_{\mathcal{D}}$  the equivalence class of a torsion pair  $(\mathcal{X}, \mathcal{Y})$ .*

- 1 *When  $(\mathcal{X}, \mathcal{Y})$  is a t-structure, then any torsion pair in  $(\mathcal{X}, \mathcal{Y})_{\mathcal{D}}$  is a t-structure.*
- 2 *When  $(\mathcal{X}, \mathcal{Y})$  is a rigid torsion pair, then any torsion pair in  $(\mathcal{X}, \mathcal{Y})_{\mathcal{D}}$  is a rigid torsion pair.*
- 3 *When  $(\mathcal{X}, \mathcal{Y})$  is a cluster tilting torsion pair, then any torsion pair in  $(\mathcal{X}, \mathcal{Y})_{\mathcal{D}}$  is cluster tilting torsion pair.*
- 4 *When  $(\mathcal{X}, \mathcal{Y})$  is a maximal rigid torsion pair, then any torsion pair in  $(\mathcal{X}, \mathcal{Y})_{\mathcal{D}}$  is a maximal rigid torsion pair.*

*Proof.* Let  $(\mathcal{X}', \mathcal{Y}')$  be the  $\mathcal{D}$ -mutation of  $(\mathcal{X}, \mathcal{Y})$ .

- 1 When  $(\mathcal{X}, \mathcal{Y})$  is a t-structure, then the only possible mutation of  $(\mathcal{X}, \mathcal{Y})$  is  $(\mathcal{X}[1], \mathcal{Y}[1])$ , the shift of the t-structure  $(\mathcal{X}, \mathcal{Y})$ . This is because the possible functorially finite rigid  $F_2$ -subcategory  $\mathcal{D}$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$  must be zero due to Remark 3.7 and Proposition 2.9(1). It is easy to see that  $(\mathcal{X}[1], \mathcal{Y}[1])$  is a t-structure.
- 2 When  $(\mathcal{X}, \mathcal{Y})$  is a rigid torsion pair, then  $I(\mathcal{X}) = \mathcal{X}$ . It follows from Theorem 3.9 that  $I(\mathcal{X}') = \mu^{-1}(I(\mathcal{X}); \mathcal{D}) = \mu^{-1}(\mathcal{X}; \mathcal{D}) = \mathcal{X}'$ . Then  $(\mathcal{X}', \mathcal{Y}')$  is a rigid torsion pair by Proposition 2.9.
- 3 When  $(\mathcal{X}, \mathcal{Y})$  is a cluster tilting torsion pair, then  $\mathcal{Y} = \mathcal{X}[1]$  (Proposition 2.9). So for  $\mathcal{Y} = \mathcal{X}[1]$ , it is easy to see  $\mathcal{Y}' = \mathcal{X}'[1]$ . Then  $(\mathcal{X}', \mathcal{Y}')$  is cluster tilting torsion pair.
- 4 This was proved in [BMV][ZZ] for  $\mathcal{D}$  being an almost complete maximal rigid subcategory, which means  $\mathcal{D}$  contains the same but one indecomposable objects (up to isomorphisms) as  $\mathcal{X}$ . Here we present the proof for the general case for the complete for the readers. Denote  $\mathcal{Y}[-1]$  by  $\mathcal{W}$ , then  $(\mathcal{X}, \mathcal{W})$  is a maximal rigid cotorsion pair. By Theorem 3.5, we have that  $(\mathcal{X}, \mathcal{W})$  is a maximal rigid cotorsion pair in  $\mathcal{U}$ . Then so is its 0-mutation  $(\mathcal{X} < 1 >, \mathcal{W} < 1 >)$ . By Proposition 4.4 in [IY],  $\mathcal{X} < 1 > = \mathcal{X}'$  and  $\mathcal{W} < 1 > = \mathcal{W}'$ . Then by Theorem 3.5 again, we have that  $(\mathcal{X}', \mathcal{W}')$  is a maximal rigid cotorsion pair in  $\mathcal{C}$ . Therefore  $(\mathcal{X}', \mathcal{Y}')$  is a maximal rigid torsion pair in  $\mathcal{C}$ .

□

In the following, we study the property of the mutation.

**Theorem 3.11.** *Let  $\mathcal{D}$  be a functorially finite rigid  $F_2$ -subcategory of  $\mathcal{C}$ ,  $(\mathcal{X}, \mathcal{Y})$  a torsion pair with core  $I$  in  $\mathcal{C}$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$  and  $(\mathcal{X}', \mathcal{Y}') = (\mu^{-1}(\mathcal{X}; \mathcal{D}), \mu^{-1}(\mathcal{Y}; \mathcal{D}))$ . Then the following holds:*

- (1). *There is a bijection between the set of isomorphism classes of indecomposable objects in  $\mathcal{X}$  and the set of isomorphism classes of indecomposable objects in  $\mathcal{X}'$ , which induces the bijection between the set isomorphism classes of indecomposable objects in  $I$  and in  $I'$  respectively.*
- (2). *When  $\mathcal{D}$  is the proper subcategory of the core  $I$ , then  $I' \neq I$ ,  $(\mathcal{X}, \mathcal{Y}) \neq (\mathcal{X}', \mathcal{Y}')$ .*
- (3). *When  $I = \mathcal{D}$ , then  $I' = I$ ,  $\mathcal{X}' \subseteq \mathcal{X}$  and  $\mathcal{Y} \subseteq \mathcal{Y}'$ .*

*Proof.* (1). We use the same notations as the above in this section. Let  $\mathcal{Z} = {}^\perp \mathcal{D}[1] = \mathcal{D}[-1]^\perp$  and  $\mathcal{U} = \mathcal{Z}/\mathcal{D}$  the triangulated quotient category with the shift  $\langle 1 \rangle$ . Since  $(\mathcal{X}', \mathcal{Y}')$  is the  $\mathcal{D}$ -mutation of  $(\mathcal{X}, \mathcal{Y})$  with the core  $I'$ ,  $\mathcal{X}'$  ( $\mathcal{Y}'[-1]$ , or  $I'$ ) is the  $\mathcal{D}$ -mutation of  $\mathcal{X}$  ( $\mathcal{Y}[-1]$ , or  $I$ , respectively). It follows that  $\overline{\mathcal{X}'} = \overline{\mathcal{X}} \langle 1 \rangle$ ,  $\overline{\mathcal{Y}'[-1]} = \overline{\mathcal{Y}[-1]} \langle 1 \rangle$ , and  $\overline{I'} = \overline{I} \langle 1 \rangle$ . Therefore there is a bijection between the sets of isoclasses of indecomposable objects in  $\overline{\mathcal{X}}$  and in  $\overline{\mathcal{X}'}$  respectively, which induces the bijection between the sets of isoclasses of indecomposable objects in  $\overline{I}$  and in  $\overline{I'}$  respectively. The lift of the first bijection above gives a bijection between the sets of isoclasses of indecomposable objects in  $\mathcal{X}$  and in  $\mathcal{X}'$  respectively, which induces the bijection (which is also the lift of the second bijection above) between the sets of isoclasses of indecomposable objects in  $I$  and in  $I'$  respectively.

(2). Suppose  $I' = I$ . Since  $I'$  is the  $\mathcal{D}$ -mutation of  $I$ , we have that  $\mathcal{D} = I$ . Otherwise, we take a non-zero object  $X$  in  $I \setminus \mathcal{D}$ . Let  $f : X \rightarrow D$  be a left  $\mathcal{D}$ -approximation, and  $X \xrightarrow{f} D \xrightarrow{g} X' \xrightarrow{h} X[1]$  the triangle which  $f$  embeds. We have that  $X' \in I' = I$ , and then  $h = 0$  since  $\text{Ext}^1(I, I) = 0$ . It follows that the triangle above splits and  $D \simeq X \oplus X'$ . Then  $X \in \mathcal{D}$ , a contradiction. Then  $\mathcal{D} = I$ , it is a contradiction. Therefore  $I' \neq I$ , and  $(\mathcal{X}', \mathcal{Y}') \neq (\mathcal{X}, \mathcal{Y})$ .

(3). Suppose  $\mathcal{D} = I$ . Then  $I' = I = \mathcal{D}$  and  $(\overline{\mathcal{X}}, \overline{\mathcal{Y}[-1]})$  is a  $t$ -structure in  $\mathcal{U}$ . It follows that  $\overline{\mathcal{X}'} = \overline{\mathcal{X}} \langle 1 \rangle \subseteq \overline{\mathcal{X}}$  and  $\overline{\mathcal{Y}'[-1]} \subseteq \overline{\mathcal{Y}[-1]} \langle 1 \rangle = \overline{\mathcal{Y}'[-1]}$ . So we have that  $\mathcal{X}' \subseteq \mathcal{X}$  and  $\mathcal{Y} \subseteq \mathcal{Y}'$ .  $\square$

A triangulated category  $C$  is called of finite type if there are only finite many indecomposable objects up to isomorphisms in  $C$ . Finite triangulated categories are studied recently in [XZ1, XZ2][Am][Koe][Kr]. Cluster categories of Dynkin quivers and stable Cohen-Macaulay categories of finite type [BIKR] provide examples of finite triangulated categories.

**Corollary 3.12.** *Let  $C$  be a finite triangulated category,  $\mathcal{D}$  be a functorially finite rigid  $F_2$ -subcategory of  $C$ ,  $(\mathcal{X}, \mathcal{Y})$  a torsion pair with core  $I$  in  $C$  satisfying  $\mathcal{D} \subset \mathcal{X} \subset \mathcal{D}[-1]^\perp$  and  $(\mathcal{X}', \mathcal{Y}') = (\mu^{-1}(\mathcal{X}; \mathcal{D}), \mu^{-1}(\mathcal{Y}; \mathcal{D}))$  with core  $I'$ . Then  $(\mathcal{X}', \mathcal{Y}') = (\mathcal{X}, \mathcal{Y})$  if and only if  $I' = I$ .*

*Proof.* The "if" part is obviously. We prove the "only if" part. Suppose  $I' = I$ . Then by Theorem 3.11(2)  $\mathcal{D} = I$ . It follows from Theorem 3.11(3) that  $\mathcal{X}' \subseteq \mathcal{X}$ . Then we have  $\mathcal{X}' = \mathcal{X}$  from the finiteness of  $C$ . Therefore  $(\mathcal{X}', \mathcal{Y}') = (\mathcal{X}, \mathcal{Y})$ .  $\square$

We note that there are two different torsion pairs with the same core in  $C$ . For example, we take  $\mathcal{X} = \text{add} M$  with  $M$  an rigid indecomposable object in any cluster category of a quiver with at least two vertices. Then  $(\mathcal{X}, \mathcal{X}^\perp)$  and  $(\mathcal{X}^\perp[-1], \mathcal{X}[1])$  are torsion pairs, their cores are  $\mathcal{X}$ .

We also note that Corollary 3.12 is not true without the condition that  $C$  is of finite type. For example we take  $C$  to be a (bounded) derived category of the abelian category  $\mathcal{A}$ . Let  $(\mathcal{X}, \mathcal{Y})$  be the torsion pair given by the standard  $t$ -structure in  $C$ , and  $\mathcal{D} = 0$ . Then the  $\mathcal{D}$ -mutation  $(\mathcal{X}', \mathcal{Y}')$  of  $(\mathcal{X}, \mathcal{Y})$  is a  $(\mathcal{X}[1], \mathcal{Y}[1])$ . Their cores are zero, but  $\mathcal{X}' \neq \mathcal{X}$  in general.

## 4 A geometric realization of mutation

In this section, we will give a geometric interpretation of mutation of torsion pairs in the cluster categories of type  $A_n$  and in the cluster categories of type  $A_\infty$ . The geometric construction of cluster categories of type  $A_n$  (or type  $A_\infty$ ) was given by Caldero, Chapton and Shiffler [CCS] (resp. by Holm and Jørgensen [HJ]). The notion of Ptolemy diagrams was introduced recently by Holm-Jørgensen-Rubey [HJR] and by Ng [Ng] as a geometric model of torsion pairs in the cluster categories of type  $A_n$  and of type  $A_\infty$  respectively. We will define the mutation of Ptolemy diagrams for each case and prove that it coincides with the mutation of torsion pairs in the cluster categories of type  $A$  ( $A_n$  or  $A_\infty$ ).

### 4.1 Type $A_n$

Let  $P_{n+3}$  be a regular convex  $(n+3)$ -gon with the vertices  $1, 2, \dots, n+3$ , labeled counterclockwise. An edge of  $P_{n+3}$  is a set of two neighboring vertices. A diagonal of  $P_{n+3}$  is a set of two non-neighboring vertices. Caldero, Chapton and Schiffler [CCS] defined a category whose indecomposable objects are diagonals of  $P_{n+3}$  and proved that this category is equivalent to the cluster category  $C_{A_n}$  of type  $A_n$  defined in [BMRRT] (the cluster categories were defined there for arbitrary acyclic quivers). Then there is a bijection between the set of diagonals of  $P_{n+3}$  and the set of isoclasses of indecomposable objects in  $C_{A_n}$ . This bijection induces a bijection between the collection of sets of diagonals of  $P_{n+3}$  and the collection of subcategories of  $C_{A_n}$ . We denote the indecomposable object in  $C_{A_n}$  corresponding to the diagonal  $\{i, j\}$  by  $M_{\{i,j\}}$  and denote the subcategory of  $C_{A_n}$  (additively) generated by indecomposable objects corresponding to elements in the subset  $\mathfrak{U}$  of diagonals of  $P_{n+3}$  by  $\mathfrak{U}'$ . The bijection above on subsets of diagonals is denoted by  $(-)'$ . It was proved recently in [HJR] that under the bijection  $(-)'$ , there is a one-to-one correspondence between the collection of Ptolemy diagrams of  $P_{n+3}$  and the torsion pairs in  $C_{A_n}$ . For convenience, we will consider the edges of the polygon  $P_{n+3}$  to be a part of a Ptolemy diagram, and  $M_{\{i,j\}}$  of an edge  $\{i, j\}$  of  $P_{n+3}$  is zero object in  $C_{A_n}$ .

We recall the definition of Ptolemy diagrams from [HJR]. Let  $\mathfrak{U}$  be a subset of diagonals of  $P_{n+3}$ .  $\mathfrak{U}$  is called a Ptolemy diagram if it satisfies the following property: when there are two crossing diagonals  $\{i, k\}$  and  $\{j, l\}$  in  $\mathfrak{U}$ , then those of  $\{i, j\}$ ,  $\{j, k\}$ ,  $\{k, l\}$ ,  $\{l, i\}$  which are diagonals are in  $\mathfrak{U}$ . See Figure 1.

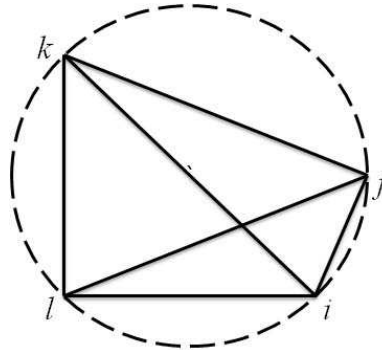


Figure 1.

We know that two diagonals  $\{i, k\}$ ,  $\{j, l\}$  cross each other if and only if  $\text{Ext}^1(M_{\{i,k\}}, M_{\{j,l\}}) \neq 0$  (see [CCS] or [HJR]). In this case  $\dim \text{Ext}^1(M_{\{i,k\}}, M_{\{j,l\}}) = 1$ . Suppose that the four vertices  $i, j, k, l$

turn clockwise order (Figure 1). Combining with the Ptolemy relation in  $P_{n+3}$  which corresponds to the exchange relations in the cluster algebra of type  $A_n$  studied in Section 12.3 of [FZ] and the equivalence of the categories of diagonals of  $P_{n+3}$  with the cluster categories of type  $A_n$  studied in [CCS], there are two non-split triangles in  $C_{A_n}$ :

$$M_{\{j,l\}} \rightarrow M_{\{i,j\}} \oplus M_{\{k,l\}} \rightarrow M_{\{i,k\}} \rightarrow M_{\{j,l\}}[1]$$

and

$$M_{\{i,k\}} \rightarrow M_{\{j,k\}} \oplus M_{\{i,l\}} \rightarrow M_{\{j,l\}} \rightarrow M_{\{i,k\}}[1].$$

By Theorem A in [HJR],  $(\mathcal{U}', \mathcal{U}'^\perp)$  is a torsion pair in  $C_{A_n}$  for any Ptolemy diagram  $\mathcal{U}$ . We denote the set of edges of  $P_{n+3}$  by  $E(P)$  and the set  $\{\alpha \in \mathcal{U} \mid \alpha \text{ crosses no diagonal in } \mathcal{U}\}$  by  $I(\mathcal{U})$ . It is easy to see that  $I(\mathcal{U})' = I(\mathcal{U}')$ . Given a subset  $\mathcal{D}$  of  $I(\mathcal{U})$ . A set  $\{i_1, i_2, \dots, i_s\}$  of integers with  $1 \leq i_1 < i_2 < \dots < i_s \leq n+3$  is called a  $\mathcal{D}$ -cell if  $\{i_1, i_2\}, \dots, \{i_{s-1}, i_s\}, \{i_1, i_s\}$  are in the set  $E(P) \cup \mathcal{D}$  and other subsets formed by two elements of  $\{i_1, i_2, \dots, i_s\}$  are not in  $\mathcal{D}$ . We call a diagonal  $\{a, b\}$  is in the interior of  $\mathcal{D}$ -cell  $\{i_1, \dots, i_s\}$  with  $i_1 < \dots < i_s$  if  $a, b \in \{i_1, \dots, i_s\}$  and  $1 < |b - a| < s - 1$ , i.e.  $\{a, b\}$  is not the edge of the  $\mathcal{D}$ -cell  $\{i_1, \dots, i_s\}$ . We note that the Ptolemy diagram  $\mathcal{U}$  of polygon  $P_{n+3}$  is divided to several  $\mathcal{D}$ -cells by diagonals in  $\mathcal{D}$ .

Now we define the  $\mathcal{D}$ -mutation of a Ptolemy diagram  $\mathcal{U}$  of the polygon  $P_{n+3}$ .

**Definition 4.1.** 1. Let  $\{i_1, i_2, \dots, i_s\}$  be a  $\mathcal{D}$ -cell with  $i_1 < i_2 < \dots < i_s$  of  $\mathcal{U}$ . For any diagonal  $\{i_k, i_l\}$  in the interior of this  $\mathcal{D}$ -cell, we define the  $\mathcal{D}$ -mutation of  $\{i_k, i_l\}$  is  $\{i_{k-1}, i_{l-1}\}$  where we take  $i_0$  to be  $i_s$ . For any diagonal  $\{a, b\}$  in  $\mathcal{D}$ , we define the  $\mathcal{D}$ -mutation of  $\{a, b\}$  is itself. The  $\mathcal{D}$ -mutation of the  $\mathcal{D}$ -cell is the set of  $\mathcal{D}$ -mutations of diagonals in this cell.  
2. The  $\mathcal{D}$ -mutation of  $\mathcal{U}$  is the union of the  $\mathcal{D}$ -mutations of its  $\mathcal{D}$ -cells.

Since every diagonal in  $\mathcal{U}$  but not in  $\mathcal{D}$  belongs only one  $\mathcal{D}$ -cell as one of its diagonals, the definition of  $\mathcal{D}$ -mutation of  $\mathcal{U}$  above is well-defined.

**Remark 4.2.** Roughly speaking, the  $\mathcal{D}$ -mutation of  $\mathcal{U}$  is a diagram obtained by the following operator: the endpoints of diagonals in  $\mathcal{U}$  which are not in  $\mathcal{D}$  are moved clockwise along the boundary of the  $\mathcal{D}$ -cell containing the diagonal to the next vertices. For example, if  $\mathcal{D}$  is empty, then the  $\mathcal{D}$ -mutation of  $\mathcal{U}$  is the Ptolemy diagram obtained by shifting clockwise the diagonals in  $\mathcal{U}$ .

**Proposition 4.3.** The  $\mathcal{D}$ -mutation of a Ptolemy diagram  $\mathcal{U}$  is also a Ptolemy diagram.

*Proof.* For any two crossing diagonals of  $\mathcal{U}$ , those two diagonals are not in  $I(\mathcal{U})$ . Then they belong to the same  $\mathcal{D}$ -cell. It follows that the  $\mathcal{D}$ -mutation of  $\mathcal{U}$  is a Ptolemy diagram.  $\square$

**Example 1.** Let  $n = 5$ , and  $\mathcal{U} = \{\{2, 7\}, \{2, 8\}, \{3, 7\}, \{3, 8\}, \{4, 6\}, \{4, 7\}, \{5, 7\}\}$  be a Ptolemy diagram of  $P_{n+3}$ , see Figure 2. Then  $I(\mathcal{U}) = \{\{2, 8\}, \{3, 7\}, \{4, 7\}\}$ .

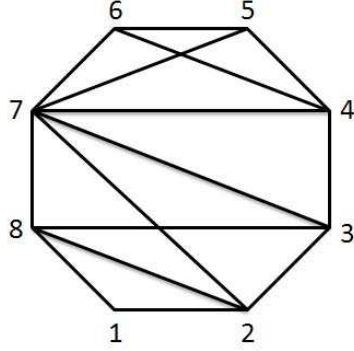


Figure 2.

Let  $\mathcal{D} = \{\{3, 7\}\}$  which is a subset of  $I(\mathfrak{U})$ . Then  $\mathfrak{U}$  divides into two  $\mathcal{D}$ -cells:  $\{1, 2, 3, 7, 8\}$ ,  $\{3, 4, 5, 6, 7\}$ . Then the  $\mathcal{D}$ -mutation of  $\mathfrak{U}$  is the following diagram, see Figure 3.

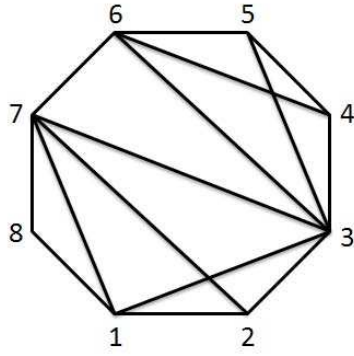


Figure 3.

Now we prove the main result in this subsection which gives a geometric realization of mutation of torsion pairs in  $C_{A_n}$ .

**Theorem 4.4.** *Under the correspondence  $(-)'$ , the  $\mathcal{D}$ -mutation of a Ptolemy diagram  $\mathfrak{U}$  coincides with the  $\mathcal{D}'$ -mutation of the torsion class  $\mathfrak{U}'$  of the torsion pair  $(\mathfrak{U}', \mathfrak{U}'^\perp)$ .*

*Proof.* Let  $\mathfrak{U}$  be a Ptolemy diagram of  $P_{n+3}$ ,  $\mathcal{D}$  a subset of  $I(\mathfrak{U})$ . Then  $\mathfrak{U}$  divides into several  $\mathcal{D}$ -cells. Let  $\{i_k, i_l\}$  be in  $\mathfrak{U}$ , not in  $\mathcal{D}$ . Then  $\{i_k, i_l\}$  belongs to a  $\mathcal{D}$ -cell, say  $\{i_1, i_2, \dots, i_s\}$ . Then we have the following triangle in  $C_{A_n}$ :

$$M_{\{i_k, i_l\}} \rightarrow M_{\{i_{k-1}, i_k\}} \oplus M_{\{i_{l-1}, i_l\}} \rightarrow M_{\{i_{k-1}, i_{l-1}\}} \rightarrow M_{\{i_k, i_l\}}[1].$$

The diagonals  $\{i_{k-1}, i_k\}$  and  $\{i_{l-1}, i_l\}$  are the edges of the  $\mathcal{D}$ -cell  $\{i_1, i_2, \dots, i_s\}$ , so they are edges of  $P_{n+3}$  or in  $\mathcal{D}$ . Then  $M_{\{i_{k-1}, i_k\}} \oplus M_{\{i_{l-1}, i_l\}} \in \mathcal{D}'$ . The diagonal  $\{i_{k-1}, i_{l-1}\}$  crosses no diagonals in  $\mathcal{D}$ , so  $\text{Ext}^1(M_{\{i_{k-1}, i_{l-1}\}}, \mathcal{D}') = 0$ . Therefore,  $M_{\{i_{k-1}, i_{l-1}\}} = \mu_{\mathcal{D}'}^{-1}(M_{\{i_k, i_l\}})$ , the mutation of object  $M_{\{i_k, i_l\}}$  in the cluster category  $C_{A_n}$ .

□

## 4.2 Type $A_\infty$

The cluster category  $C_{A_\infty}$  was introduced in [HP]. It is the orbit category  $D^f(mod\Gamma)/\tau^{-1}[1]$ . Here  $\Gamma$  is a quiver of type  $A_\infty$  with zigzag orientation and  $[1]$  and  $\tau$  are the Auslander-Reiten translate and the shift functor of the finite derived category  $D^f(mod\Gamma)$ . Recall that the Auslander-Reiten quiver of  $C_{A_\infty}$  is  $\mathbb{Z}A_\infty$ . See Figure 4.

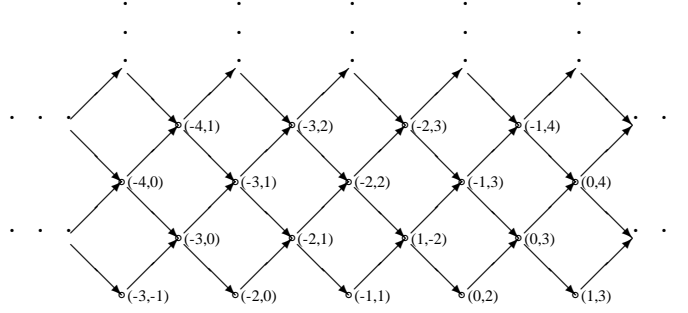


Figure 4. The AR-quiver of  $C_{A_\infty}$ .

The action of the shift functor  $[1]$  is  $(a, b)[1] = (a - 1, b - 1)$ .

Let  $P_\infty$  be a  $\infty$ -gon with the vertices labeled by integers. An edge of  $P_\infty$  is a set  $\{a, b\}$  of integers with  $|b - a| = 1$ . An arc of  $P_\infty$  is a set  $\{a, b\}$  of integers with  $|b - a| \geq 2$ . For any arc  $\{a, b\}$ , we denote the indecomposable object in  $C_{A_\infty}$  corresponding to the pair  $(\min\{a, b\}, \max\{a, b\})$  by  $M_{\{a, b\}}$ . Sometimes we use  $(\min\{a, b\}, \max\{a, b\})$  to denote  $M_{\{a, b\}}$  for simplicity if no confusion arising. Then the map  $\{a, b\} \mapsto M_{\{a, b\}}$  induces a bijection between the set of arcs of  $P_\infty$  and the set of isoclasses of indecomposable objects in  $C_{A_\infty}$  (see [HJ]). This bijection induces a bijection between the collection of sets of arcs of  $P_\infty$  and the collection of subcategories of  $C_{A_\infty}$ . This bijection is denoted by  $(-)'$ . We denote the subcategory of  $C_{A_n}$  (additively) generated by indecomposable objects corresponding to elements in the subset  $\mathcal{U}$  of arcs of  $P_\infty$  by  $\mathcal{U}'$ . Two arcs  $\{a, b\}$  and  $\{c, d\}$  are said to cross each other if we have either  $a < c < b < d$  or  $c < a < d < b$ . We recall some notion from [HJ].

**Definition 4.5.** ([HJ] Definition 3.2) Let  $A$  be a set of arcs. If for each integer  $n$  there are only finitely many arcs in  $A$  which end in  $n$ , then  $A$  is called *locally finite*. If  $n$  is an integer such that  $A$  contains infinitely many arcs of the form  $\{m, n\}$  (resp.  $\{n, p\}$ ), then  $n$  is called a *left* (resp. *right*) *fountain* of  $A$ . If  $n$  is both a left and a right fountain of  $A$ , then it is called a *fountain*.

It was proved recently in [Ng] that for a subset  $\mathcal{U}$  of arcs,  $(\mathcal{U}', \mathcal{U}'^\perp)$  is a torsion pair in  $C_{A_\infty}$  if and only if  $\mathcal{U}$  satisfies the following two conditions:

- (i). If for each pair of crossing arcs  $\{a, b\}$  and  $\{c, d\}$  in  $\mathcal{U}$ , those of the pairs  $\{a, c\}$ ,  $\{c, b\}$ ,  $\{b, d\}$  and  $\{a, d\}$  which are arcs belong to  $\mathcal{U}$  (see Figure 5);
- (ii). Each right fountain of  $\mathcal{U}$  is a fountain.

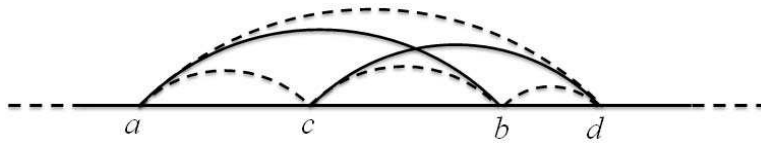


Figure 5.

We will call the subset  $\mathcal{U}$  a Ptolemy diagram of  $P_\infty$ . We know that two arcs  $\{a, b\}$  and  $\{c, d\}$  cross if and only if  $\dim \text{Ext}(M_{\{a,b\}}, M_{\{c,d\}}) = 1$  (see Lemma 3.6 in [HJ]). We can determine that the non-split triangles between  $M_{\{a,b\}}$  and  $M_{\{c,d\}}$ . For convenience, we will use  $M_{\{a,a+1\}}$  to denote the object corresponding to the edge  $\{a, a+1\}$  of  $\infty$ -gon. It is zero object in  $C_{A_\infty}$ .

**Lemma 4.6.** *Suppose that  $\{a, b\}$  and  $\{c, d\}$  are two arcs of  $P_\infty$  with  $a < c < b < d$ . Then there are two non-split triangles in  $C_{A_\infty}$  between  $(a, b)$  and  $(c, d)$ :*

$$(a, b) \rightarrow (a, d) \oplus (c, b) \rightarrow (c, d) \rightarrow (a, b)[1]$$

and

$$(c, d) \rightarrow (a, c) \oplus (b, d) \rightarrow (a, b) \rightarrow (c, d)[1].$$

*Proof.* The first triangle is from the AR-quiver of  $C_{A_\infty}$  straightly. It is a non-split triangle. We show the existence of the second one. Note that there are two non-split triangles in  $C_{A_\infty}$  which are from the AR-quiver of  $C_{A_\infty}$ :

$$(c, b+1) \rightarrow (c, d) \rightarrow (b, d) \rightarrow (c, b+1)[1],$$

$$(c-1, b)[-1] \rightarrow (a, c) \rightarrow (a, b) \rightarrow (c-1, b).$$

Applying  $\text{Hom}((b, d), -)$  to the last triangle, we have that the following exact sequence

$$\text{Hom}((b, d), (a, c)) \rightarrow \text{Hom}((b, d), (a, b)) \xrightarrow{f} \text{Hom}((b, d), (c-1, b)).$$

When  $d-b \geq 2$ , we have that  $a+1 < b < b+1 < d$  and  $a+1 < c+1 \leq b < d$ . We have that  $(a+1, b+1)$  and  $(b, d)$  cross,  $(a, b) = (a+1, b+1)[-1]$ ;  $(a+1, c+1)$  and  $(b, d)$  don't cross. By Lemma 3.6 in [HJ], it follows that  $\text{Hom}((b, d), (a, b)) \cong k$  and  $\text{Hom}((b, d), (a, c)) = 0$ . Therefore  $f$  is injective and then bijective because  $\dim_k \text{Hom}((b, d), (c-1, b)) \leq 1$ . When  $d-b = 1$ ,  $(b, d)$  is a zero object in  $C_{A_\infty}$ , then  $f$  is also bijective. So any map from  $(b, d)$  to  $(c-1, b)$  factors through  $(a, b)$ . But  $(c, b+1)[1] = (c-1, b)$ , then there is the following commutative diagrams

$$\begin{array}{ccccccc} (c, b+1) & \rightarrow & (c, d) & \rightarrow & (b, d) & \rightarrow & (c, b+1)[1] \\ \parallel & & \downarrow & & \downarrow & & \parallel \\ (c-1, b)[-1] & \rightarrow & (a, c) & \rightarrow & (a, b) & \rightarrow & (c-1, b). \end{array}$$

By Lemma 2.2 in [XZ1], we get the second triangle in the lemma. It is a non-split triangle since  $(c, d) \oplus (a, b) \not\cong (a, c) \oplus (b, d)$ .  $\square$

We denote the set of edges of  $P_\infty$  by  $E(P_\infty)$  and the set  $\{\alpha \in \mathcal{U} \mid \alpha \text{ crosses no arc in } \mathcal{U}\}$  by  $I(\mathcal{U})$ . It is easy to see that  $I(\mathcal{U})' = I(\mathcal{U}')$ . Given a subset  $\mathcal{D}$  of  $I(\mathcal{U})$ . It is not difficult to see that  $\mathcal{D}$  contains at most one fountain. By Theorem 2.2 and Theorem 2.3 in [Ng], we have that  $\mathcal{D}'$  is functorially finite if and only if  $\mathcal{D}$  is locally finite or has a fountain. Under this condition on  $\mathcal{D}$ , a finite set  $\{i_1, i_2, \dots, i_s\}$  with  $i_1 < i_2 < \dots < i_s$  of integers is called a finite  $\mathcal{D}$ -cell if  $\{i_1, i_2\}, \dots, \{i_{s-1}, i_s\}, \{i_s, i_1\}$  are in the set  $E(P_\infty) \cup \mathcal{D}$  and other subsets of the form  $\{i_k, i_l\}$  of  $\{i_1, i_2, \dots, i_s\}$  are not in  $\mathcal{D}$ ; 'an infinite set  $\{i_t \mid t \in \mathbb{Z}\}$  of integers where  $i_t < i_s$  for  $t < s$  is called an infinite  $\mathcal{D}$ -cell if  $\{i_t, i_{t+1}\}$  is in the set  $E(P_\infty) \cup \mathcal{D}$  for any  $t \in \mathbb{Z}$  and other subsets of the form  $\{i_k, i_l\}$  of  $\{i_t \mid t \in \mathbb{Z}\}$  are not in  $\mathcal{D}$ .

We call an arc  $\{a, b\}$  is in the interior of a finite  $\mathcal{D}$ -cell  $\{i_1, \dots, i_s\}$  with  $i_1 < \dots < i_s$  (resp. an infinite  $\mathcal{D}$ -cell  $\{i_t \mid t \in \mathbb{Z}\}$  where  $i_t < i_s$  if  $t < s$ ) if  $a, b \in \{i_1, \dots, i_s\}$  and  $1 < b-a < s-1$  (resp.  $a, b \in \{i_t \mid t \in \mathbb{Z}\}$  and  $b-a \geq 2$ ).



**Proposition 4.7.** *Let  $\mathcal{U}$  be Ptolemy diagram of  $P_\infty$  and  $\mathcal{D}$  be a subset of  $I(\mathcal{U})$  satisfying that it is locally finite or has a fountain. Then  $\mathcal{U}$  divides into several  $\mathcal{D}$ -cells by arcs in  $\mathcal{D}$  (maybe infinitely many).*

*Proof.* We need to prove that any arc  $\{a, b\}$  ( $a < b$ ) in  $\mathcal{U}$  but not in  $\mathcal{D}$  belongs to a  $\mathcal{D}$ -cell. We will construct a  $\mathcal{D}$ -cell containing the arc  $\{a, b\}$ . There is the maximal integer  $a_1$  such that  $a < a_1 \leq b$  and  $\{a, a_1\}$  belongs to  $\mathcal{D} \cup E(P_\infty)$ . By induction, we can find  $a_i$ ,  $2 \leq i \leq m$  such that  $a_i$  is the maximal integer satisfying that  $a_{i-1} < a_i \leq b$  and  $\{a_{i-1}, a_i\}$  belongs to  $\mathcal{D} \cup E(P_\infty)$  for any  $2 \leq i \leq m$ , and  $a_m = b$ .

If there is not any arc  $\{c, b\} \in \mathcal{D} \cup E(P_\infty)$  with  $c \leq a$ , then  $b$  is not a left fountain of  $\mathcal{D}$  and then is not a right fountain of  $\mathcal{D}$ . Then there is the maximal integer  $b_1$  such that  $\{b, b_1\} \in \mathcal{D} \cup E(P_\infty)$ . We denote  $b$  by  $b_0$ . By induction, either we can find  $b_i$ ,  $0 \leq i \leq n$ , such that  $b_i$  is the maximal integer satisfying that  $\{b_i, b_{i+1}\} \in \mathcal{D} \cup E(P_\infty)$  for  $0 \leq i \leq n-1$  and there is an arc  $\{c, b_n\} \in \mathcal{D} \cup E(P_\infty)$  with  $c \leq a$  but any arc  $\{c', b_i\}$  is not in  $\mathcal{D} \cup E(P_\infty)$  for  $c' \leq a$  and  $0 \leq i \leq n-1$ , or we can find  $b_i$ ,  $i \geq 0$ , such that  $b_i$  is the maximal integer satisfying that  $\{b_i, b_{i+1}\} \in \mathcal{D} \cup E(P_\infty)$  for  $i \geq 0$  and any arc  $\{c, b_i\}$  is not in  $\mathcal{D} \cup E(P_\infty)$  for any  $c \leq a$  and  $i \geq 0$ .

For the first case, we take  $c$  to be the maximal integer such that  $\{c, b_n\} \in \mathcal{D} \cup E(P_\infty)$  and  $c < a$ . We denote  $c$  by  $c_0$ . By the same method above, we can find  $c_i$ ,  $1 \leq i \leq l$ , such that  $c_i$  is the maximal integer satisfying that  $\{c_{i-1}, c_i\} \in \mathcal{D} \cup E(P_\infty)$  and  $c_i \leq a$ , and  $c_l = a$ . We claim that  $\{c, c_1, \dots, c_{l-1}, a, a_1, \dots, a_{m-1}, b, b_1, \dots, b_n\}$  is a  $\mathcal{D}$ -cell. By construction, any set consisting of two neighboring numbers in the set  $\{c, c_1, \dots, c_{l-1}, a, a_1, \dots, a_{m-1}, b, b_1, \dots, b_n\}$  belong to  $\mathcal{D} \cup E(P_\infty)$  where we view  $b_n, c$  are also the neighboring numbers. Let  $\{x, y\}$  be an arc consisting of two non-neighboring numbers in the set  $\{c, c_1, \dots, c_{l-1}, a, a_1, \dots, a_{m-1}, b, b_1, \dots, b_n\}$ . If  $x, y$  belong to the same set  $\{c, c_1, \dots, c_l\}$ ,  $\{a, a_1, \dots, a_m\}$ , or  $\{b, b_1, \dots, b_n\}$ , then this arc is not in  $\mathcal{D}$  by the choose of  $c_i$ ,  $a_i$  or  $b_i$ . If  $x$  is in the set  $\{c, c_1, \dots, c_{l-1}\}$  or  $\{b_1, \dots, b_n\}$ ,  $y$  is in the set  $\{a_1, \dots, a_{m-1}\}$ , then this arc cross the arc  $\{a, b\}$ . Then  $\{x, y\}$  is not in  $\mathcal{D}$ . If  $x$  is in the set  $\{c, c_1, \dots, c_l\}$ ,  $y$  is in the set  $\{b, b_1, \dots, b_n\}$ , then  $\{x, y\}$  is not in  $\mathcal{D}$  by the choose of  $b_n$  and  $c$ . This proves our claim.

For the second case,  $a$  is not a fountain of  $\mathcal{D}$ . Otherwise there is an arc  $\{a, b'\} \in \mathcal{D}$  with  $b' > b$ . If  $b' = b_i$  for some  $i$ , this is a contradiction. If  $b_i < b' < b_{i+1}$  for some  $i$ , then  $\{a, b'\}$  cross  $\{b_i, b_{i+1}\}$ , a contradiction. So there is the minimal integer  $a_{-1}$  such that  $\{a_{-1}, a\} \in \mathcal{D} \cup E(P_\infty)$ . As the same reason,  $a_{-1}$  is not a fountain of  $\mathcal{D}$ . Then by induction, we can find  $a_i$ ,  $i \leq -1$ , such that  $a_i$  is the minimal integer satisfying that  $\{a_i, a_{i+1}\} \in \mathcal{D} \cup E(P_\infty)$  for  $i \leq -1$ . Similar as the first case, we have that  $\{a_i, i \leq m-1\} \cup \{b_i, i \geq 0\}$  is a  $\mathcal{D}$ -cell. It is an infinite  $\mathcal{D}$ -cell. □

Now we define the  $\mathcal{D}$ -mutation of Ptolemy diagram  $\mathcal{U}$  of the polygon  $P_\infty$ .

**Definition 4.8.** 1. Let  $\{i_1, i_2, \dots, i_s\}$  be a finite  $\mathcal{D}$ -cell with  $i_1 < i_2 < \dots < i_s$ . For any arc  $\{i_k, i_l\}$  in the interior of this  $\mathcal{D}$ -cell, we define the  $\mathcal{D}$ -mutation of  $\{i_k, i_l\}$  is  $\{i_{k-1}, i_{l-1}\}$  where we take  $i_0$  to be  $i_s$ . Let  $\{i_t \mid t \in \mathbb{Z}\}$  be an infinite  $\mathcal{D}$ -cell where  $i_t < i_s$  for  $t < s$ . For any arc  $\{i_k, i_l\}$  in the interior of this  $\mathcal{D}$ -cell, we define the  $\mathcal{D}$ -mutation of  $\{i_k, i_l\}$  is  $\{i_{k-1}, i_{l-1}\}$ . For any arc  $\{a, b\}$  in  $\mathcal{D}$ , we define the  $\mathcal{D}$ -mutation of  $\{a, b\}$  is itself. The  $\mathcal{D}$ -mutation of the  $\mathcal{D}$ -cell is the set of  $\mathcal{D}$ -mutations of diagonals in this cell.

2. The  $\mathcal{D}$ -mutation of  $\mathcal{U}$  is the union of the  $\mathcal{D}$ -mutations of its  $\mathcal{D}$ -cells.

Since every arc in  $\mathcal{U}$  but not in  $\mathcal{D}$  belongs to the interior of only one (finite or infinite)  $\mathcal{D}$ -cell, the definition of  $\mathcal{D}$ -mutation of  $\mathcal{U}$  above is well-defined.

**Example 2.** Let  $\mathfrak{U} = \{\{1, 3\}, \{1, 4\}, \{2, 4\}, \{1, 5\}\} \cup \{\{t, 1\} \mid t \leq -1\}$ , see Figure 6. Then  $I(\mathfrak{U}) = \{\{1, 4\}, \{1, 5\}\} \cup \{\{t, 1\} \mid t \leq -1\}$ .

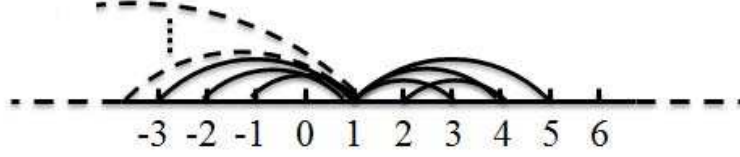


Figure 6.

Let  $\mathcal{D} = \{\{1, 5\}\}$  which is a subset of  $I(\mathfrak{U})$  and is locally finite. Then  $\mathfrak{U}$  divides into two  $\mathcal{D}$ -cells: a finite  $\mathcal{D}$ -cell  $\{1, 2, 3, 4, 5\}$  and an infinite  $\mathcal{D}$ -cell  $\{t \mid t \leq 1 \text{ or } t \geq 5\}$ . Then the  $\mathcal{D}$ -mutation of  $\mathfrak{U}$  is the following diagram, see Figure 7.

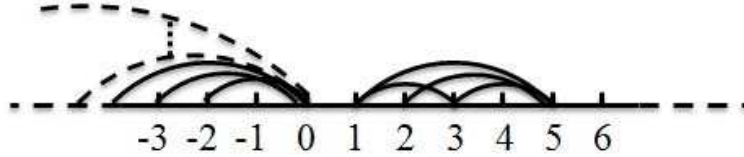


Figure 7.

**Theorem 4.9.** Under the correspondence  $(-)'$ , the  $\mathcal{D}$ -mutation of a Ptolemy diagram  $\mathfrak{U}$  of  $P_\infty$  coincides with the  $\mathcal{D}'$ -mutation of the torsion class  $\mathfrak{U}'$  of the torsion pair  $(\mathfrak{U}', \mathfrak{U}'^\perp)$ .

*Proof.* Let  $\mathfrak{U}$  be a Ptolemy diagram of  $P_\infty$ ,  $\mathcal{D}$  a subset of  $I(\mathfrak{U})$  satisfying that  $\mathcal{D}$  is locally finite or has a fountain. Then  $P_\infty$  divides into several  $\mathcal{D}$ -cells. Let  $\{i_k, i_l\}$  be in  $\mathfrak{U}$ , not in  $\mathcal{D}$ . Then  $\{i_k, i_l\}$  belongs to the interior of a  $\mathcal{D}$ -cell, say  $\{i_1, i_2, \dots, i_s\}$  if it is finite or  $\{i_t \mid t \in \mathbb{Z}\}$  if it is infinite, then we have the following triangles in  $C_{A_\infty}$  by Lemma 4.6:

$$M_{\{i_k, i_l\}} \rightarrow M_{\{i_{k-1}, i_k\}} \oplus M_{\{i_{l-1}, i_l\}} \rightarrow M_{\{i_{k-1}, i_{l-1}\}} \rightarrow M_{\{i_k, i_l\}}[1]$$

where we take  $i_0$  to be  $i_s$  when the  $\mathcal{D}$ -cell is finite.

The arcs  $\{i_{k-1}, i_k\}$ ,  $\{i_{l-1}, i_l\}$  are in  $E(P_\infty) \cup \mathcal{D}$ . Then  $M_{\{i_{k-1}, i_k\}} \oplus M_{\{i_{l-1}, i_l\}}$  belongs to  $\mathcal{D}'$ . The arc  $\{i_{k-1}, i_{l-1}\}$  crosses no diagonals in  $\mathcal{D}$  since it is in a  $\mathcal{D}$ -cell, so  $\text{Ext}^1((i_{k-1}, i_{l-1}), \mathcal{D}') = 0$ . Therefore,  $M_{\{i_{k-1}, i_{l-1}\}} = \mu_{\mathcal{D}'}^{-1}(M_{\{i_k, i_l\}})$ , the mutation of object  $M_{\{i_k, i_l\}}$  in the cluster category  $C_{A_\infty}$ .  $\square$

From the theorem above and Theorem 3.18 [Ng], we have that the  $\mathcal{D}$ -mutation of a Ptolemy diagram of  $P_\infty$ .

**Corollary 4.10.** If  $\mathfrak{U}$  is a Ptolemy diagram of  $P_\infty$ , then so is the  $\mathcal{D}$ -mutation of  $\mathfrak{U}$ .

## References

- [Am] C. Amiot. On the structure of triangulated categories with finite many indecomposables. *Bull. Soc. math.France.* **135**(3), 435-474, 2007.
- [ASS] I.Assem, D.Simson and A.Skowronski. Elements of the representation theory of associative algebras. Vol. 1. Techniques of representation theory, London Mathematical Society Student Texts, 65. Cambridge University Press, Cambridge, 2006.
- [BBD] A.A.Beilinson, J.Bernstein and P.Deligne. Faisceaux pervers. *Astérisque.* **100**, 1981.
- [BIRS] A.Buan, O.Iyama, I.Reiten and J.Scott. Cluster structure for 2-Calabi-Yau categories and unipotent groups. *Compo. Math.* **145**(4), 1035-1079, 2009.
- [BIKR] I.Burban, O.Iyama B.Keller and Idun Reiten. Cluster tilting for one-dimensional hypersurface singularities. *Advances in Mathematics,* **217**(6), 2443-2484, 2008.
- [BMV] A.Buan, R.Marsh and D.Vatne. Cluster structure from 2-Calabi-Yau categories with loops. *Math.Zeite.*, **265**(4), 951-970, 2010.
- [BMRRT] A.Buan, R.Marsh, M.Reineke, I.Reiten and G.Todorov. Tilting theory and cluster combinatorics. *Advances in Math.* **204**, 572-618, 2006.
- [BR] A.Beliannis and I.Reiten, Homological and homotopical aspects of torsion theories, *Memoirs of the AMS* **883** (2007), 426-454.
- [CCS] P.Caldero, F.Chapoton and R.Schiffler. Quivers with relations arising from clusters ( $A_n$  case). *Transactions of the AMS.* **358**, 1347-1364, 2006.
- [D] S.D.Dickson, A torsion theory for abelian category, *Trans. AMS.* **121** (1966), 223-235.
- [FZ] S.Fomin and A.Zelevinsky. Cluster Algebras I: Foundations. *Jour. Amer. Math. Soc.* **15**, no. 2, 497-529, 2002.
- [H] D.Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras. London Mathematical Society, LNM 119, Cambridge, 1988.
- [HJ] T.Holm, P.Jørgensen. On a triangulated category which behaves like a cluster category of infinite Dynkin type, and the relation to triangulations of the infinity-gon. To appear in *Math. Zeite*, see also arXiv:0902.4125.
- [HJR] T.Holm, P.Jørgensen, M.Rubey. Ptolemy diagrams and torsion pairs in the cluster category of Dynkin type  $A_n$ . To appear in *Jour. Alg. Comb.* see also arXiv:1010.1184.
- [IY] O.Iyama and Y. Yoshino. Mutations in triangulated categories and rigid Cohen-Macaulay modules. *Invent. Math.* **172** (2008), no.1, 117-168.
- [Ke1] B.Keller. Triangulated orbit categories. *Documenta Math.* **10**, 551-581, 2005.
- [Ke2] B.Keller. Cluster algebras, quiver representations and triangulated categories. In *Triangulated categories*, volume 375 of LMS Lecture Notes Ser., 375, Cambridge, Pages 76-160. Cambridge Univ. Press, Cambridge, 2010. See also arXiv: 0807.1960.

- [Ke3] B.Keller. Triangulated Calabi-Yau categories, Trends in Representation Theory of Algebras (Zurich) (A. Skowronski, ed.), European Mathematical Society, 2008, pp. 467C489.
- [KR] B.Keller and I.Reiten. Cluster-tilted algebras are Gorenstein and stably Calabi-Yau. Adv. Math. 211, 123-151, 2007.
- [Koe] C.Koehler. Thick subcategories of finite algebraic triangulated categories. arXiv:1010.0146.
- [Kr] H.Krause. Report on locally finite triangulated categories. arXiv:1101.3233.
- [KeV] B.Keller and D.Vossieck, Aisles in derived categories. Buu.Soc.Math. Belg. 40,239-253, 1988.
- [KZ] S.Koenig, and B.Zhu. From triangulated categories to abelian categories— cluster tilting in a general framework. Math. Zeit. 258, 143-160, 2008.
- [MP] R. Marsh and Y. Palu. Coloured quivers for rigid objects and partial triangulations: The unpunctured case. Preprint. arXiv:1012.5790.
- [N] H.Nakaoka. General heart construction on a triangulated category (I): unifying t-structures and cluster tilting subcategories. To appear in Appl. Categ. Theory, see also arXiv:math.RT/0907.2080.
- [Ng] Puiman Ng. A characterization of torsion theories in the cluster category of Dynkin type  $A_\infty$ . Preprint. arXiv:1005.4364.
- [Re] I.Reiten. Cluster categories. arXiv:1012.4949, to appear in Proceedings of ICM 2010.
- [XZ1] J. Xiao and B.Zhu. Relations for the Grothendieck groups of triangulated Categories. Jour. of Algebra, **257**, 37-50, 2002.
- [XZ2] J. Xiao and B.Zhu. Locally finite triangulated categories, Jour. Algebra, **290**, 473-490, 2005.
- [ZZ] Y.Zhou and B.Zhu. Maximal rigid subcategories in 2–Calabi-Yau triangulated categories. arXiv:1004.5475.